MATH 5012

Exercise 8

(1) Take $n=1$ and $U=\mathbb{R}$. Find the distributional derivatives of the following functions:

(a) $\frac{d^2}{dx^2}|x|$,

(b) $\frac{d}{dx}H(x)$, $H(x)=\begin{cases}1, & x>0,\\ 0, & x\le 0\end{cases}$ (Heaviside function)

(c) $\frac{d}{dx}|\cos x|$, $\frac{d^2}{dx^2}|\cos x|$.

(2) Distributions $\Lambda_j \in \mathcal{D}(U)$ is said to converge to $\Lambda \in \mathcal{D}(U)$ if $\Lambda_j\varphi \to \Lambda\varphi$, $\forall \varphi \in \mathcal{D}(U)$. Show that the trigonometric series $\sum_{-\infty}^{\infty} c_n e^{inx}$, $|c_n| \le C_1|n|^{\ell} + C_2$, converges to a distribution in $\mathcal{D}'(\mathbb{R})$.

(3) Consider the 2π-periodic function

$$F(x) = \frac{x(2\pi - x)}{4\pi}, \quad 0 \le x \le 2\pi.$$

Show that

$$\sum_{-\infty}^{\infty} \delta(x-2\pi n) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{inx}.$$

Hint: Use the Fourier expansion

$$F(x) = \frac{\pi}{6} - \frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{n^2} e^{inx}.$$

(4) Let $H_1(x,y) = H(x)H(y)$ where H is in 1(b). Verify that $\frac{\partial^2}{\partial x \partial y} H_1(x,y) = \delta$ (δ at $(0,0)$).

(5) Let $f \in BV(a,b)$ and left-continuous. ($BV(a,b)$ means $BV[c,d]$, $\forall [c,d] \subset (a,b)$.) Show that

$$D\Lambda_f = \Lambda_\mu,$$

where the measure μ satisfies $\mu[c,d] = f(d) - f(c)$. Deduce that $D\Lambda_f = \Lambda_{Df}$ iff $f \in AC(a,b)$.

(6) A distribution has order N if $\forall K \subset\subset U$, $\exists C_K$ s.t.

$$|\Lambda\varphi| \leq C_K \|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}_K.$$

(a) Determine the order of δ and the Heasivide function H in $\mathcal{D}'(-1,1)$.

(b) Show that the distribution

$$\Lambda\varphi = \sum_{k=1}^{\infty} \left(\frac{d^k\varphi}{dx^k}\right)\left(\frac{1}{k}\right), \quad \varphi \in \mathcal{D}(-1,1),$$

has no finite order.